Topic 2: Time Value of Money and Present Value

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In this note, we discuss the concept of **Present Value** (PV). We begin by considering a simple example of choosing when to receive some money from a Grandparent. We use this example to motivate the claim that there is a **time value** assigned to money, that then naturally motivates the idea of present value. This allows us to provide a formal definition of PV. We then show how to calculate PV for a given income stream, using the definition. Okay, let's get cracking!

1 Money from Grandma

Suppose that every year, your Grandma gives you $50 for your birthday, but she really doesn't mind when she gives it to you (i.e. if it's on your birthday or not). Instead, she lets you choose when to receive the $50. Suppose it is January 1st; when do you want to receive the money? Right now of course!!

Whilst receiving $50 from Grandma today will mean I receive the same amount of money from Grandma in 2021 as if I received the $50 in, say, December, most people prefer to receive the $50 now. Why? Well typically we explain this behaviour by assuming that **money has some time value**; in other words, I prefer to have money now, rather than later.

Suppose now that your Grandma wishes to encourage you to defer gratification. She tells you, you can have $49.50 now (January 1st), or $50 on December 31st. Which would you choose? Money now please Grandma!!

What this example is showing is that not only do I prefer money now, **I will actually take less money now, rather than waiting for more money later**. Suppose that, understanding this, Grandma keeps lowering the money now; suppose she now offers you \$40 now or \$50 in December. Will you accept? Maybe! Let's suppose she keeps lowering the money value until you are **indifferent** between the money now and the money later. For sake of example, let's suppose that this amount is \$30. **Well, this is exactly the present value of \$50 on December 31st**! The **present value** of a future payment is just the value we would consider to be its equal if we were to receive that value instead, but right now.

Note that the future value of the payment from Grandma is still \$50; I will receive \$50 from Grandma if I wait until December 31st. But **how I value that payment presently is at only \$30**.

2 Formalising the intuition

So in the last section we motivated the idea that money now is worth more than money later. We also showed that, for a value of money in the future, there was a value of money **today** that was **less** than that future value, that we nonetheless valued equally to that future value. How do we formalise this notion? We begin by defining **discount rates**. We then show how discount rates help us to build a formal definition of PV.

2.1 Discount Rates

When talking about Grandma, I relied on your intuition to motivate you to agree that \$50 today is better than \$50 in a years time. How do we formalise that idea? Well, one way of doing it is to say, look: if I give you \$50 today, you can place that 50% in a bank account, with an interest rate of r. In a year's time, that \$50 will actually be worth $(1 + r)\$50$. So receiving \$50 in the future is actually worth **less** to me in real terms than receiving that \$50 now.

What value of money today would mean that the real value of money today were the same as the \$50 in the future? Well, if I start with $\$X$ dollars, I will have $(1 + r)\$X$ at the end of the year. So I just need to find the value of X such that $(1 + r)\$X = \$50$. Some simple algebra shows me that this value for X is $\frac{\$50}{1+r}$. In other words, the value of the \$50 in a years time has to be **discounted by** $\frac{1}{1+r}$ in order for me to understand how much I should value it **presently**.

When we talk about discount rates in the context of PV, **the value we're talking about is** r. This value gives me the **opportunity cost** of receiving money later versus receiving it today; if I receive money later, **I lose the opportunity to invest it**. Given that I could be better off if I were to have this opportunity, I would prefer to have the money now!

Now suppose that Grandma is being especially wicked, and is trying to induce you to wait a whole two years before you receive your birthday money. Then, following the same reasoning as before, how much would she have to lower your offer of money today so that the value of the future money is equal to that of the present, which we'll call $\$X$ again?

Well, if invest $\$X$ today, I will have $(1+r)\$X$ next year. And if I invest that again in the following year, I'll have $(1+r)\left((1+r)\$X\right) = (1+r)^2\$X$. So what is the value of X that makes me indifferent to money now versus money in two years time? It will be the X such that $(1+r)^2\$X = \$50$. In other words, $X = \left(\frac{1}{1+r}\right)^2 \$50$.

So we see that a way of calculating the **present value** of a future payment of \$50 that I receive in t years time is given by:

$$PV(\$50 \text{ in } t \text{ years time}) = \left(\frac{1}{1+r}\right)^t \$50$$

With this expression in mind, let's move to a formal definition of PV.

2.2 The centrality of opportunity

The discount rate that we employ in finance is always related to some notion of **opportunity cost**. The idea is that there are many places that we can place our money, and by placing it in one area, we are by definition eliminating the *opportunity* of investing it elsewhere.

Note that this is different from the economic notion of discounting. In standard economics, we often assume some **time preference** that is parameterised by β. Whilst this concept can coincide with the financial notion of discounting, which occurs when $\beta = \frac{1}{1+r} \leftrightarrow \beta(1+r) = 1$, *they are not necessarily the same.*

In the example above, by deferring receipt of cash to a future date, we are *implicitly* choosing to hold an asset that delivers zero returns for holding it. This means we **cannot** hold an alternative asset that *does* deliver a positive return.

So, how do we choose which is the appropriate 'opportunity cost' to use to calculate the discount rate? Opportunities have two components: risk and reward. When we're estimating the value of *our* project, we want to compare it to alternative opportunities that have *similar levels of risk and reward*. Then we can say, 'well the interest rate on those opportunities is roughly r, so we should use that rate to discount this project'. Thus, what PV is doing is comparing a project *relative to other similarly risky/rewardy projects.*

2.2.1 Discount rates vs Discount Factors

This is just a small note to emphasise that the value, r, is the discount *rate*, whereas the value we use to discount payoffs, $\frac{1}{1+r}$ is the discount *factor*. Remember this!

2.3 Building a Formal Definition of PV

So we saw in the previous example that a way to work out the present value of a future income is to **discount** that income using the discount rate r, so that we account for the **opportunity cost** of not being able to invest our money and make a return. This basic idea is essentially the definition of present value. All we will do now is generalise our example; instead of just one payment of \$50, we'll allow there to be many payments, at different times in the future. Formally:

Definition. *The **Present Value** of a stream of payments, $Y = \{y_0, y_1, y_2, y_3, ...\}$ is equal to the **sum** of the **discounted** values of those payments, where the **discount rate**, r, accounts for the **opportunity cost** of not being able to invest the money and earn a return:*

$$PV(Y) = \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t y_t$$

What is this definition telling me? It's telling me that, for a given stream of payments, (e.g. $\{\$10, \$24, \$15, ...\}$), I would value those payments **the same** as receiving $\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t y_t$ right now! I have converted a string of future payments into a single **present value** of those payments.

3 Using this Definition

In this section we'll work through a couple of examples of how to make use of this definition when actually looking at income streams. We begin by imagining an offer to pay you \$10 a year, every year, forever. We then suppose I offer to pay you \$10 a year, every year, for the next 20 years.

3.1 An infinite stream of Hamiltons

Figure 1: The man, the myth, the legend: Alexander Hamilton!

I'm a huge fan of the musical Hamilton, as I'm sure we all are. I'm such a huge fan, in fact, that I insist on keepin all of my earnings in \$10 bills, which I covet in my minuscule New York City apartment. As an Economics PhD student who therefore earns an enormous salary[1], I therefore have a hell of a lot of Hamiltons.

After hearing that you too are a fan of Hamilton, I decide to make you an offer. I will share my enormous stock of Hamilton \$10 bills with you, giving you one a year forever (and to your kids, and their kids, and their kids, and so on). But I won't do this for free! If I did, **I would never be satisfied**. I want you to give me some money upfront. How much is fair, do you think?

Suppose that the interest rate I can access at my local Chase Bank is 2%. Well, let's use this and apply our formula! We just saw that for a stream of payments, $Y = \{y_0, y_1, y_2, ...\}$, the present value of those payments was:

$$PV(Y) = \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t y_t$$

Let's populate with our numbers from our example:

$$\begin{aligned} PV(Hamilton) &= \sum_{t=0}^{\infty} \left(\frac{1}{1+0.02} \right)^t 10 \\ &= 10 \sum_{t=0}^{\infty} \left(\frac{1}{1.02} \right)^t \end{aligned}$$

In Math class, we should have all seen how we calculate the value of this sum, right?

[1] Please read with sarcasm

We know that for an **infinite geometric sequence** (i.e. something that looks like $\sum_{n=0}^{\infty} a^n$), then, so long as $0 < a < 1$:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

Applying this to our example:

$$\begin{aligned}
PV(Hamilton) &= 10 \sum_{t=0}^{\infty} \left(\frac{1}{1.02}\right)^t \\
&= 10 \left(\frac{1}{1 - \frac{1}{1.02}}\right) \\
&= 10 \left(\frac{1}{\frac{0.02}{1.02}}\right) \\
&= 10 \left(\frac{1.02}{0.02}\right) \\
&= \$510
\end{aligned}$$

So, given the opportunity cost of not being able to invest the money in the bank, the present value of the stream of Hamiltons is \$510. In other words, you should be indifferent between receiving \$510 now, or \$10 a year every year for the rest of eternity.

So, what would be a good price to offer me for my Hamiltons? Well, if you offer me \$510, then this is going to be **exactly equal to the loss of the \$10 every year in present value terms**. I should be indifferent between the \$510 now or keeping my Hamiltons. If you offer me less than \$510, I won't accept, and if you offer me more than \$510 I will for sure accept, but you will be paying more than you need to! So \$510 seems a fair price. So, cough up!

3.2 A finite stream of Hamiltons

The last example was a little hokey... I can't really promise to give someone a Hamilton a year for the rest of time. Let's suppose that I offer you something more sensible; a Hamilton a year for the next twenty years. **Would that be enough?** Well, let's see how much you can offer me in return!

Note that we can use the same definition as before; i.e. there is an infinite income stream $Y = \{y_0, y_1, y_2, ...\}$, and the present value of those payments is:

$$PV(Y) = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t$$

But now, all $y_t = 0$ for all $t > 20$. In other words:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t = \sum_{t=0}^{20} \left(\frac{1}{1+r}\right)^t y_t$$

In our example, $y_t = 10$ for all t between 0 and 20. So the present value of the finite stream of Hamiltons is:

$$\begin{aligned} PV(Hamilton) &= \sum_{t=0}^{20} \left(\frac{1}{1+0.02}\right)^t 10 \\ &= 10 \sum_{t=0}^{20} \left(\frac{1}{1.02}\right)^t \end{aligned}$$

Now we have a **finite** sum. Maybe we remember the formula for the sum of a finite geometric sequence[2]?

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

So, we apply this as before:

$$\begin{aligned} PV(Hamilton) &= 10 \sum_{t=0}^{20} \left(\frac{1}{1.02}\right)^t \\ &= 10 \left(\frac{1-\left(\frac{1}{1.02}\right)^{21}}{1-\frac{1}{1.02}}\right) \\ &= 10(17.35...) \\ &= \$173.50 \end{aligned}$$

So as you'd expect, this is worth a lot less. Hamiltons forever versus Hamiltons for 20 years is a big difference! Of course, in both cases you would have to **wait for it**.

[2] I literally never do, I had to look it up again while writing this note...

4 Application to Corporate Finance

In the preceding sections we used a couple of silly examples to motivate the notion of money having time value, and to construct a measure that values a stream of payments over time as a one-off payment today. These silly examples actually map quite neatly onto real-world financial instruments that are relevant to us in corporate finance! To make this clear, consider the following table, where we look at three example instruments:

Financial Product	**Simple Definition**	**Present Value**
Annuity	A promise to pay a series of fixed payments, D_t, over a pre-specified length of time, T, at regular intervals.	$PV(Annuity) = \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t D_t$
Corporate Bond	A promise *by a company* to pay a series of fixed payments, D_t, where D_t can be 0, over a pre-specified length of time, T.	$PV(Bond) = \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^t D_t$
Perpetuity	A promise to pay a series of fixed payments, D_t, *indefinitely.*	$PV(Perpetuity) = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t D_t$

Let's look a little closer at bonds to hammer this point home.

4.1 Valuing Bonds

What is a bond? A bond is in essence a type of loan. When I issue a bond, I am asking you to lend me money over a specific time horizon, with the promise of interest for that loan. Bonds are made up of three key metrics:

- The **principal**
- The **coupon payments**
- The *maturity*

The principal denotes the amount of money I am loaning. The coupon payments denote the interest I promise to pay for you. The maturity denotes the number of periods until I pay you back. Bonds are issued by companies, governments, and even pop stars[3]. To illustrate how bonds work with an example, suppose I offer

[3] David Bowie sold bonds that gave access to future payments from royalties and live performances, known as **Bowie Bonds**

you a bond today with a principal of \$100, with coupon payments of \$8.50, that matures in December 2025 (four year maturity). Note we can express the coupon payments as a rate of 8.5%. Your first coupon payment will be in December 2022, then December 2023, and so on. Then the payoffs associated with that bond are:

Cash Payments

2022	*2023*	*2024*	*2025*
\$8.50	\$8.50	\$8.50	\$108.50

Note that in 2025, I pay you a coupon payment *and* the value of the principal. Thus, the overall payments that accrue to you are \$8.50 $\times$ 4 = \$34, plus the return of the principal that you gave me to buy the bond. Suppose that this bond has roughly the same risk-reward profile as alternative bonds that offer a coupon rate of 3%. What is the PV of this bond?

$$\begin{aligned}
PV &= \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t \\
&= \sum_{t=0}^{3} \left(\frac{1}{1.03}\right)^t y_t \\
&= \left(\frac{1}{1.03}\right) \times 8.5 + \left(\frac{1}{1.03}\right)^2 \times 8.5 + \left(\frac{1}{1.03}\right)^3 \times 8.5 + \left(\frac{1}{1.03}\right)^4 \times 108.5 \\
&= \$120.44
\end{aligned}$$

4.1.1 Price of a bond

Note that we haven't yet mentioned the *price* of this bond. The price will not necessarily be equal to the principal, as the price is typically determined by demand and supply, not the issuer. The process by which this occurs involves a sort of auction that occurs at the point of issuance. If this bond looks particularly attractive, the price to buy it may be quite a bit more than the \$100 principal.

Bond prices are usually represented by the *percentage of the principal*. That is, if this bond were to cost \$110, the price would be quoted as 110%. Suppose that the principal was \$500 instead of \$100, and the price of the bond were \$550—then the bond price would also be quoted at 110%.

4.1.2 Return of a bond

Suppose that the bond we discussed above were available at a price of \$120.44, or 120.44%. One question we might have is, *what is the return on holding this bond*? In other words, how much interest will I receive if I spend \$120.44 on this bond? To calculate that, we can do one of two things:

- Work out the total return across the entire period.
- Work out the *annual rate of return* that this bond implies.

In the case of the former, this is easy: we just take the total payments, \$134, and divide them buy the total outlays, \$120.44, and minus 1. In this example, and in this case, the return is roughly 11%. For the latter case, what we're interested in is the value of y that solves this expression:

$$120.44 = \frac{8.5}{1+y} + \frac{8.5}{(1+y)^2} + \frac{8.5}{(1+y)^3} + \frac{108.5}{(1+y)^4}$$

Why? What is this expression telling us? The left hand side is how much I paid for the bond. The right hand side, without the $\frac{1}{1+y}$ terms are the payoffs. By scaling these payoffs by a discount factor that would *reduce* these payouts such that the return would be zero, we are *implicitly* calculating what the *actual* annual returns are! The value of y that solves this expression is called the *yield to maturity* —it captures the *annual rate of return* that holding the bond would deliver if you held it all the way to maturity[4].

How do we find this value of y? Note that we have to solve a polynomial of 4 degrees... not easy! Unlike in the quadratic case, *no general solution for such an expression exists*. To solve it, we have to rely on **numerical approximations** —we plug numbers in a computer until they work! In this particular case, however, we know what y should be... that's because the expression on the right hand side is essentially the same as the PV calculation we performed above! It must be the case that $y = 0.03$. However, if the price of the bond were \$115, for example, it would not be so obvious what y should be (it's around 4.34%).

Sidenote: Suppose the bond cost \$125; should you buy it? In this scenario, the cost of the bond *exceeds its present value*. Another way of saying this is that its *Net*

[4]The 'to maturity' part captures the fact that many people do not hold bonds to maturity, but instead sell them in secondary markets

Present Value is negative. We'll talk more about NPV next class, but hopefully you can see why we might argue that the bond is *overpriced* at \$125, and that buying it is not such a good idea.

4.2 Changing the Maturity Length

In the example above, we focused on a very simple case of a single bond, with a defined maturity. We were able to calculate the present value of that bond using the formula we described above, and assuming a discount rate appropriate to that bond. We were also able to calculate the *yield to maturity* of a bond, if we know its price, its coupon payments, and its principal. If the price is less than or equal to the PV, we suggested that this makes the bond a good investment.

Typically, bond issuers will not restrict themselves to issuing a single type of bond. The US government, for example, issues bonds across a very broad range of maturities. Three key examples of these bonds are:

- **Treasury Bills**: bonds with a maturity of one year or less. They have *zero coupon payments* meaning the government only pays back the face value.
- **Treasury Notes**: bonds with a maturity of 2, 3, 5, 7, or 10 years, with coupon payments every six months, and sold in increments of \$100.
- **Treasury Bonds**: bonds with a maturity of 20 or 30 years, also paying coupon payments every six months.

How does varying the maturity affect a bond? Suppose we keep the coupon payments and the principal fixed. Will the present value change? We should expect so right? Consider two bonds, both paying \$8.50 coupons, with a principal of \$100, where one has a two year maturity (call it Bond A), and the other a three year maturity (call it Bond B). Suppose that other bonds in the market with similar risk have an interest rate of 3%. Then:

$$PV(A) = \frac{1}{1.02} \times 8.5 + \frac{1}{(1.02)^2} 108.5 \approx \$112.62$$

$$PV(B) = \frac{1}{1.02} \times 8.5 + \frac{1}{(1.02)^2} 8.5 + \frac{1}{(1.02)^3} 108.5 \approx \$118.75$$

So clearly the present value of these bonds are different. Suppose that their prices reflect their present value; i.e. Bond A costs \$112.62, and Bond B costs \$118.75.

Are their *yields to maturity* different? Using the method described above, then y^A and y^B are the solutions to the following two equations:

$$112.62 = \frac{1}{1+y^A} \times 8.5 + \frac{1}{(1+y^A)^2} 108.5$$

$$118.75 = \frac{1}{1+y^B} \times 8.5 + \frac{1}{(1+y^B)^2} 8.5 + \frac{1}{(1+y^B)^3} 108.5$$

Given the PV exercise we just performed, we know it must be the case that $y^A = y^B$. So, changing maturity shouldn't change the yield... right?

4.2.1 The Term Structure

It is very easy to find the yields of US government bonds, so we can check to see if our hypothesis is correct. Suppose we were to plot the maturity of the bonds on the x-axis, and the yields of these different maturity bonds on the y-axis. This presentation is called a **yield curve**. Figure 2 shows the most recent yield curve for US government bonds.

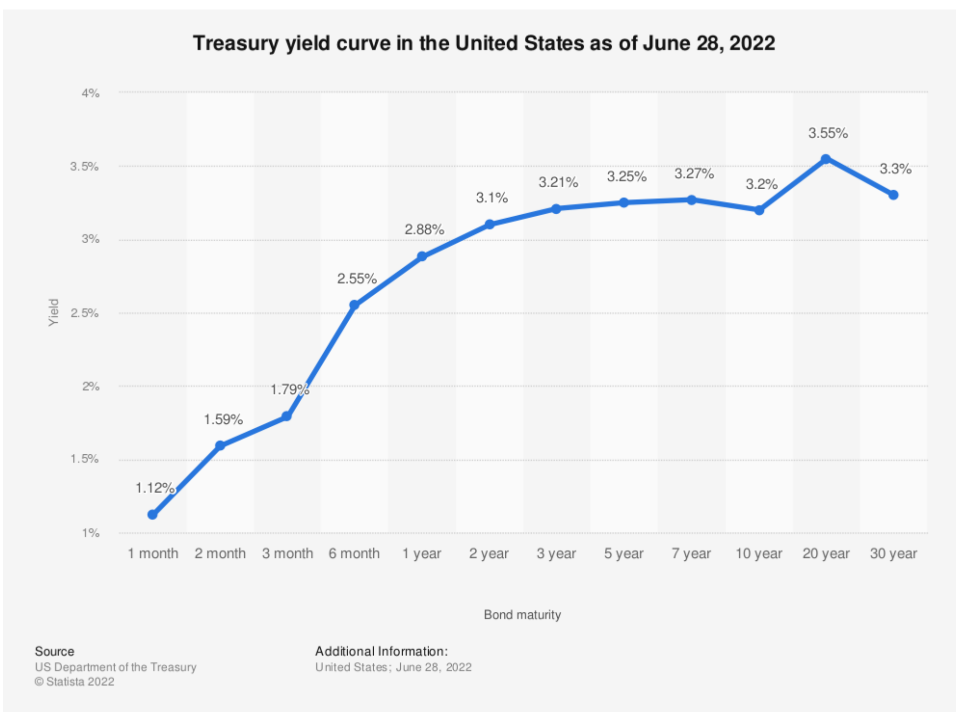

Figure 2: Yield curve for US Government Bonds as of June 2022

What do we see? **A clear upward slope**. This is the classic shape that the US yield curve takes, and seems to contradict our finding in the last section. So where did we go wrong?

Note that throughout, we've assumed that the opportunity cost, r, is fixed across time. Is this a sensible assumption? Is the opportunity cost of investing in a government bond the same during a boom vs. a recession? Surely not! The quality and quantity of opportunities is constantly changing over time. So, instead of using a single discount rate, r, perhaps it makes more sense to use a changing rate, r_t

Let's go back to our example from before, and now suppose that $r_1 = 0.02$, $r_2 = 0.03$, and $r_3 = 0.04$. Intuitively what does this mean? It means we think that there will be *greater opportunities in the future* for projects that are *similarly risky* to that of our bond. What does modifying the maturity structure do now? Again, we begin by calculating the PV:

$$PV(A) = \frac{1}{1.02} \times 8.5 + \frac{1}{(1.03)^2} 108.5 \approx \$110.61$$

$$PV(B) = \frac{1}{1.02} \times 8.5 + \frac{1}{(1.03)^2} 8.5 + \frac{1}{(1.04)^3} 108.5 \approx \$112.80$$

Now we calculate the yield to maturity —remember, that this number gives the *annual* return, so will only be a single number! The yields can be found by solving the following two equations:

$$110.61 = \frac{1}{1 + y^A} \times 8.5 + \frac{1}{(1 + y^A)^2} 108.5$$

$$112.80 = \frac{1}{1 + y^B} \times 8.5 + \frac{1}{(1 + y^B)^2} 8.5 + \frac{1}{(1 + y^B)^3} 108.5$$

The solution to these two equations are: $y^A = 0.030$ and $y^B = 0.039$. So, when opportunity costs are expected to increase, the yield to maturity goes up!

With this in mind, perhaps we can give an intuitive explanation as to why the US yield curve is upward sloping? Can you think of a reason? Here are two clues: (i) the geometric average of US growth rates over the last 20 years was roughly 3.9%; and (ii) the geometric average of the annual returns on the S&P 500 over the last twenty years was roughly 6.8%. The answer is that *most people expect the US to be more productive in the future*. If the US will be more productive in the future, I need to discount future payments *by even more*, as the opportunities of cash available to

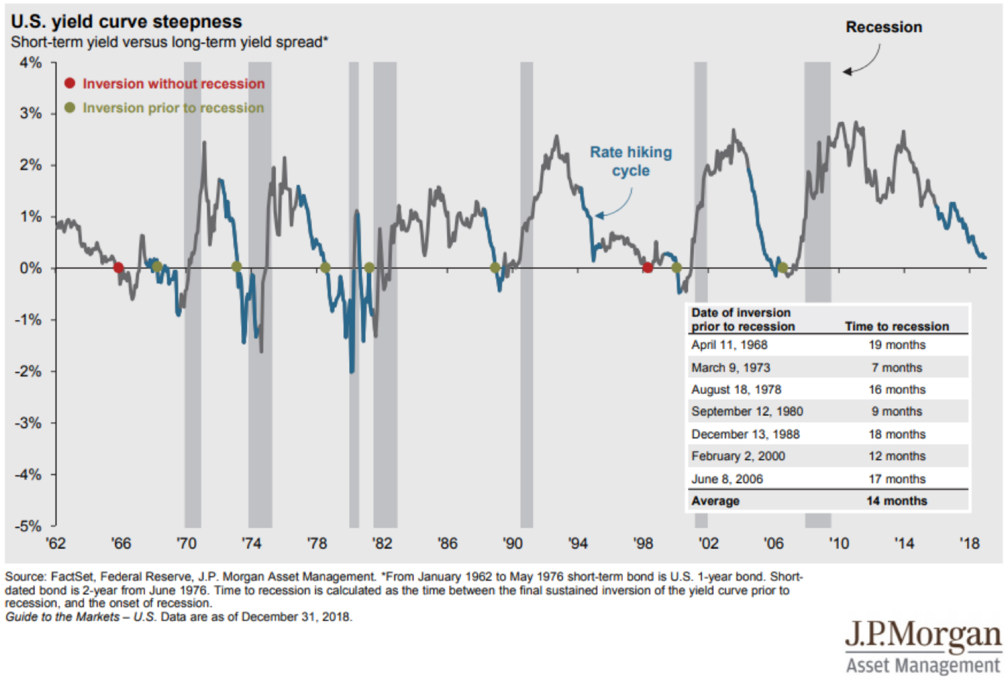

Figure 3: Shows the difference(*spread*) between the yield to maturity of 10-year vs. 2-year government bonds. Negative values imply an inverted yield curve

be in those periods *is commensurately higher*!

One result that springs out of this finding is that *inverted* US yield curves (i.e. one that goes *down* rather than *up*) are often a good predictor of a coming recession. In other words, the market sees the opportunities as being *lesser* in the future. To show the power of this result, take a look at Figure 3. This figure shows the difference (also known as the **spread**) between the yields of 10 year maturity US bonds vs. 2 year maturity bonds. Note that if this difference is positive, we have an upward sloping yield curve, and if it is negative, we have a downward sloping yield curve. We can clearly see that the yield curve 'inverts' (i.e. the spread becomes negative) shortly before recessions. Cool huh?

5 Summary

In this note, we used a simple example of money from Grandma to motivate the notion of present value. We formalised this notion using the opportunity cost of money, noting that we don't earn interest on money later. We then showed how we can apply this notion to a stream of payments, and turn that stream into a single number, that tells me how much I value the stream compared to a single payout right now. We then explored how PV can be applied to bonds. We then built an understanding of how bonds work, how the PV relates to their return, plus a few key facts/features related to bonds.